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ANALYTICAL MECHANICS.

A Treatise on Analytical Mechanics. By Bartholomew Price, M.A., F.R.S., F.R.A.S., Sedleian Professor of Natural Philosophy, Oxford. Vol. II. Dynamics of a Material System. Second Edition. (Oxford: Clarendon Press, 1889)

A SECOND title-page describes the present work as volume iv. of "A Treatise on Infinitesimal Calculus," so that Prof. Bartholomew Price's well-known four volumes may be taken to represent the curriculum of the Infinitesimal Calculus and its applications for the mathematical student at Oxford.

To one accustomed to the style of the text-books in use at Cambridge, the contrast is very striking; the Oxford student is much to be envied for the leisurely and luxuriant way in which the subject is here presented, which follows on the lines of Lagrange and Laplace, and utilizes all the resources of analysis. A student who has been through the present work will be prepared to appreciate the purely geometrical form in which the Newtonian methods, insisted upon at Cambridge, would present some of the theorems in a more fundamental and incisive form; but to our mind the Cambridge system is inferior, which ostensibly insists on the purely geometrical methods before allowing the student to make use of the power of analysis.

Although Newton claims to be one of the inventors of this Calculus, and must have employed its methods in the discovery of his theorems, yet he carefully covered up all traces of the analytical scaffolding, and exhibited a theorem in the "Principia," like a Greek temple, in pure geometrical form.

His influence on his successors was too great when they attempted to follow in the same lines, with the consequence that our insular school of mathematics lagged hopelessly in rear of Continental progress.

Although prescribed as the text-book at Cambridge, the "Principia" is not studied in the original Latin, as Newton wrote it, from one end to the other; but the student makes use of commentaries and selections, which, in accordance with the regulations, he professes to appreciate and apply, before knowing even by sight the supposed mystifying symbols of dy/dx and $\int ydx$.

We might as well send out our soldiers armed with muzzling loading guns, or even bows and arrows, to meet a continental army equipped with the most recent inventions of magazine rifles and breech-loading artillery.

Thus the late R. A. Proctor could write that, although a wrangler, he knew nothing of the Differential Calculus till some time afterwards, when he had to pick it up of himself; however, by a recent regulation, only passed a few weeks ago, a most stupendous change has been made in the Mathematical Tripos, by prescribing a certain very elementary course of Analytical Geometry and the Calculus in the First Three Days.

At Cambridge the large number of candidates for mathematical honours acts as a check to change; and as the same papers have to serve for such widely different classes as the wranglers and the junior optimes, it may happen that a candidate who merely writes out book-work will beat a better mathematician who is tempted to try the difficult questions.

The number of students in mathematics at Oxford is much smaller, and the standard for honours is higher; so that we can take this treatise on Infinitesimal Calculus and contrast it with the extracts from Newton's "Principia," to illustrate the relative standards.

Under the enthusiastic influence of a Sylvester we may see the mathematical school at Oxford the first in this country, as it was two hundred years ago, in the days of Wren, Wallis, Keill, and the founders of the Royal Society, which had its origin in Oxford.

At the outset of the Dynamics of a Material System in space, it is necessary to discuss a number of theorems in solid geometry on the distribution of principal axes and the associated theorems of confocal quadrics (chapter i.); also the kinematics of a rigid body, involving the composition and revolution of angular velocities, and the transformation of co-ordinate axes (chapter ii.).

The author could simplify the distinction between the two systems of rectangular axes by adopting Maxwell's comparison with the screw, right-handed or left-handed. All specifications of rotation as clock-wise, or counter-clock-wise, are ambiguous; because the direction changes as we pass from one side to the other of the clock face. Standing at dusk about a quarter of a mile from a windmill, nearly in the plane of the sails, it is possible by a slight mental effort to change the apparent direction of rotation, and back again, as often as we please.

The author does not permit himself the use of the elliptic functions; or else he would have found the wonderful chapter i., t. ii., of Halphen's "Fonctions Elliptiques" of great service in giving the representation of the cosines of the angles which a movable straight line or a movable set of three rectangular axes makes with three fixed rectangular axes. Much of the subsequent work on Euler's three angles, the integration of his equations of motion, and of the spherical pendulum, &c., could be completed and the integrations effected by the use of Halphen's formulas.

Dynamics proper is introduced in chapter iii., where D'Alembert's principle is employed to establish the equations of motion of a material system, with the subsequent corollaries of the independence of the motions of translation and rotation, and the principles of the conservation of momentum and energy.

It is more the fashion now to dispense with D'Alembert's principle, and to refer immediately to Newton's third law of motion; still, D'Alembert's principle, although a mere corollary, states the thing in such a way as to lead immediately to the formation of the six equations of motion; and by stating it in such a manner as to reduce all dynamical principles to a statical form—"the reversed effective forces and the impressed forces form a system in equilibrium, the internal cohesive forces (stresses) being in equilibrium among themselves"—it was formerly considered that a simplification was effected.

But Maxwell, in his "Matter and Motion," by considering Newton's third law of motion as merely the definition of a *stress*, has been able to restate all the theorems involved in D'Alembert's principle in a few simple sentences, and in a much more convincing form.

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It is a pity that the term *vis viva* has been allowed to remain in this last edition, and that it was not entirely replaced by *kinetic energy:* the contrasted term *vis mortua* has been dead for a long time, and *vis viva* should have followed long ago.

The transformations of the dynamical equations into the Lagrangian and Hamiltonian forms are introduced at rather an early stage; and the subject is resumed in the last chapter x., on "Theoretical Dynamics," written by the late Prof. W. F. Donkin. These transformations are merely analytical illustrations of the change of independent variables, the form of the equations depending on whether we express the kinetic energy in terms of the generalized velocities or the generalized momenta.

A clear and expressive notation, somewhat in the style of that found necessary in Thermodynamics, would make these equations more intelligible and convincing; but in any case, the application to definite problems, especially where the geometrical constraints present any peculiarity, is so difficult and refined, that these equations are dangerous weapons to put into the hands of any but advanced students.

The principles of Least Action and of Least Constraint are also introduced here by the author; interesting verifications are thus afforded of well-known problems; but these principles again would not be employed for choice; and although the author pleads in their favour, we think it should not be forgotten that the principle of Least Action was employed to bolster up the Corpuscular Theory of Light.

Newton's principle of mechanical similitude, in the next section, is, however, of great practical importance, and we see its application in the constantly increasing size of our bridges, ships, and guns. In its particular application to naval architecture, a corollary goes by the name of Froude's law (also enunciated by Reech), which asserts that in similar vessels run at speeds proportional to the square root of the length or the sixth root of the displacements, the resistances are as the displacements; and thus the naval architect is able to infer, from the known performance of a ship or a model, what to expect on a different scale. When we make, in any two similar machines of the same material, the velocities in the ratio of the square root of the linear dimensions, we ensure in this manner that the stress per unit area in the material is the same, and thus the two machines are equally strong; so that this law of corresponding speed is most useful in the practical application of Newton's law of similitude.

A valuable section on Units, No. 9, points out that there are only two systems which need be considered: the British foot-pound-second (F.P.S.) system, and the metric centimetre-gramme-second (C.G.S.) system. The author's numbers for the conversion of one system into the other are not exactly according to the latest determinations; thus it is more accurate to make 1 metre = 39·37079 inches, and 1 foot = 30·4794 centimetres. The metre was originally designed so that the kilometre should be the centesimal minute of latitude, for use in navigation; but taking the sexagesimal minute of latitude as 6080 feet, the Admiralty standard, then the above figures make the length of the earth's quadrant 10,007 kilometres, instead of 10,000, as designed. It has been decided, however, for electrical purposes that 10^9 centimetres should be called a *quadrant*, although about 0·07 per cent. out.

Recent redeterminations of the weight of a metre cube of water, and of the volume of 10 gallons or 100 pounds of water, made with the greatest care, have revealed perceptible discrepancies with former estimates; so that the definition of the kilogramme as a decimetre cube of pure water at its maximum density must be considered a purely academic definition, and not sufficiently precise for legal purposes; the ultimate appeal being to the lump of platinum preserved in the Conservatoire des Arts et Métiers.

A Committee of the British Association is at present engaged in attempting to fill up the gaps in our dynamical terminology: the author introduces the *dyne* and *erg*, due to a former Committee, but not the *kine*, *spoud*, *bole*, and *barad*, recently settled upon as names for the C.G.S. units of velocity, acceleration, momentum, and pressure. The C.G.S. units are too minute for practical purposes, so that electricians now employ the *joule*, of 10^7 ergs, and the *watt* as the volt-ampere, or power doing one joule per second—units based really upon the commercial units of the metre and kilogramme, instead of the centimetre and the gramme. These microscopic units were adopted by the original Committee apparently merely to gratify the fad of making $W = sV$, instead of $1000sV$

The astronomical unit of mass is defined in § 143; but if it is difficult to measure the volume of a kilogramme of water, the probable error in the determination of this astronomical unit of mass is immensely greater; so that to our mind this unit had better be discarded, and the gravitation constant introduced into the equations, using its provisional value, $10^{-8} \times 6{\cdot}48$ C.G.S. units (Everett, "Units and Physical Constants," § 72).

Chapter iv. discusses the equations of motion of a rigid body expressed in terms of angular velocities and their increments, &c. The author adopts various illustrative methods, but to our mind the simplest procedure is to establish the general equations, with the usual notation $\dot{h}_1 - h_2\theta_3 + h_3\theta_2 = \mathrm{L}, \ldots$; and then Euler's equations, &c., follow as particular cases. By adding the terms due to the employment of a movable origin we obtain the form of the Hamiltonian equations required in the discussion of the motion of a body moving in a liquid; and here is a good opportunity for the introduction of Dr. Routh's principle of the Ignoration of Co-ordinates, required to complete the theory of the generalized equations of motion.

Prof. Price could make a very useful book for students of elementary mathematics by taking out and printing separately the part on uniplanar motion and its illustrative examples (chapter v.): the complication of the subject of rigid dynamics is more than doubled when we consider motion in three dimensions; but in two dimensions the subject is within the grasp of most students, who will thus acquire a good working knowledge sufficient for most purposes. At the outset the determination of simple moments of inertia is required, and this involves a knowledge of integration; so that a student, untrained in the Calculus, can make very little headway. It is a pity that the lack of the slight knowledge of integration required for this purpose prevents most of our students from going on to the real study of the pendulum, the motion of the wheel and wheeled carriage, and of the ballistic pendulum. Prof. Price calls the inventor Captain Robins; but, according to the preface of his "Mathematical Tracts," Robins was of Quaker extraction (like so many other students and inventors of warlike instruments), and his only military employment was as chief engineer of the East India Company, in planning and carrying out their fortifications.

In chapter vi. the rotation of a rigid body about a fixed point is discussed, with applications to the three important problems of motion under no forces with Poinsot's geometrical representation, the motion of the top or gyrostat, and the precession and nutation of the earth's axis. These problems illustrate very strikingly the great increase in complication when we go from plane motion to motion in space. The figure of the herpolhode, on p. 251, shows points of inflexion; but, as the author mentions in § 295, these points of inflexion cannot exist in Poinsot's herpolhode. An elegant geometrical demonstration is given on p. 379 of Sylvester's extension of Poinsot's representation, where confocals to the momental ellipsoid are made to roll upon parallel planes; and now it is possible in certain corresponding herpolhodes for points of inflexion to make their appearance; the analytical and geometrical discussion of this problem has engaged the attention of de Sparre and Hess.

We mentioned at the outset that the author did not permit himself the use of elliptic functions; but apparently he could not resist the temptation of introducing them in the complete solution of Poinsot's motion, the simplicity and elegance of the representation being so great. In the separating case, when the modulus of the elliptic functions becomes unity, the introduction of the corresponding hyperbolic functions would have exhibited an analogous symmetry.

By considering the elliptic functions as defined by plane pendulum motion, some of the results in the motion of the top or gyrostat could have been exhibited by comparison with a plane pendulum; but it must be confessed that the simplicity is not maintained when we investigate the projection of the motion on a horizontal plane, without we introduce functions invented by Hermite, of a higher degree of complication.

In the discussion of precession and nutation, a simplification can be introduced by making use of the observed fact in determining the latitude, that the deviation of the axis of rotation from the axis of figure, although certainly existing, is quite inappreciable in the case of the earth; so that the axes of figure, of rotation, and of angular momentum may be taken as coincident. With this approximation the pole of the earth follows a point 90° in longitude behind the sun or moon with a certain velocity; and now the rest of the calculation of precession and nutation becomes a kinematical problem.

Chapter vii. discusses interesting and important problems of small oscillations and of bodies rolling on each other, *e.g.* of a billiard ball on the table; and chapter viii., on relative motion, is important as showing how far we are justified in applying our dynamical equations to the problems going on around us, considering that they take place on the surface of the earth, which is moving in a complicated manner in space. The corresponding elementary discussion in Maxwell's "Matter and Motion," on the ideas of relative motion, and the modification of the principles of dynamics to make them rigorous, is well worth attention at this point.

The deviation from the vertical of a body let fall down a deep mine, of a projectile from the vertical plane of fire, and the rotation of the plane of oscillation of Foucault's pendulum, are discussed as illustrations of the influence of the earth's rotation in modifying a dynamical question; but considering how slight a disturbing cause, such as a current of air, would be sufficient to mask the effect, we believe that these effects have not yet really been observed.

In Foucault's pendulum a very slight jockeying can make the thing go as we wish; while with artillery fire at long ranges the disturbing cause of deviation or drift quite overpowers any deviation due to the rotation of the earth. Theoretically, Foucault's pendulum, if set swinging in a plane through the rising moon, should continue to follow the moon; and roughly speaking, a shot fired at the rising moon should keep moving in the moving vertical plane through the moon, and would thus fall to one side of its original vertical plane of fire; in a range of twelve miles, and a time of flight of one minute, this deflection would, in the latitude of Shoeburyness, amount to about 71 yards, out of about 1000 yards observed average lateral deviation.

A few simple problems on the vibration of elastic threads and plates are given in chapter ix.; and chapter x., as already mentioned, is occupied by Prof. Donkin's contribution on Theoretical Dynamics.

Throughout the work good collections of illustrative examples are introduced, to test the student in his grasp of the principles given immediately before. If we might make a slight criticism, we should suggest the introduction of some arithmetical exercises on these problems, taken from examples in real life; for, as Sir William Thomson insists, it is from arithmetical applications that the student obtains a real grasp of dynamics; the examples given here only testing his algebraical and geometrical power.

In conclusion, we congratulate the student of mathematics at Oxford on the possession of such an admirable text-book, fully brought up to date in the latest developments.

A. G. GREENHILL.

ANNALS OF THE ROYAL BOTANIC GARDEN, CALCUTTA.

Annals of the Royal Botanic Garden, Calcutta. Vol. I. Appendix—(1) "Some New Species of Ficus from New Guinea," by George King, F.R.S., &c., Superintendent of the Royal Botanic Garden, Calcutta. (2) "On the Phenomena of Fertilization in *Ficus Roxburghii*, Wall," by D. D. Cunningham, F.L.S., &c., Surgeon-Major, Bengal Army. (Bengal Secretariat Press, 1889.)

ABOUT a dozen new species of Ficus are added here to Dr. King's valuable monograph of the figs of the "Indo-Malayan and Chinese countries," which occupies the whole of the first volume of the "Annals." It may be remembered that Dr. King proposed a modified classification of the species of *Ficus*, based upon characters indicating, in his view, the direction of evolution in the genus, beginning with a small group having pseudo-hermaphrodite flowers (*Palæomorphe*),